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IS "THE IDEAL FILTER" REALLY IDEAL: THE USAGE OF FREQUENCY FILTERING AND SPURIOUS CYCLES

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Abstract
In this paper the filter of Hodrick and Prescott, the band-pass filter of Baxter and King, "the ideal filter" and the first-differencing are analyzed. It is shown that ideal filters are able to produce spurious cycles not only when a unit root is present in the data but also when the root is less than unity. The main critiques of first-differencing are outlined and further analyzed. The conclusion is that differencing and more generally methods in conformity with the underlying properties of non-stationary series are the preferable choice. At the same time the application of ideal filters and their approximations (including HP and BP) is very dangerous and their usage is to be avoided.

JEL Classification: C22

Keywords: Frequency filtering, Hodrick-Prescott filter; Band-Pass Filter, Spurious Cycles

1. Introduction

One major point in the analysis of cyclical fluctuations is trend elimination, especially when attention is focused on the business cycle (Westlund and Ohlen, 1991, p. 484; Zarnowitz and Ozyildirim, 2002, p.2) and growth cycles (Zarnowitz, 1992, Zarnowitz and Moor, 1986, p.9). Many authors mention that economic time series are both fluctuating and trending, i.e. they are non-stationary.

Stock and Watson applied different methods for trend removal in analysis of the US business cycle. That includes differencing, the Butterworth filter and Baxter and

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King filter (Stock and Watson, 1989, 1990, 1996). The Hodrick-Prescott filter is very popular: Hamori and Kitisaka (1997), Serletis and Krause (1996), Pakko (1997), Olekalns (2001), Gregory, Head and Raynauld (1997). The filter suggested by Baxter and King was used by Mills (2000, 2001), Basu and Taylor (1999), Hornstein (2000), Wynne and Koo (2000). Westlund and Ohlen (1991) applied linear detrending and Beveridge-Nelson decomposition.

Generally speaking there are two main approaches to trend removal. The first one is to model the tendency with an econometric model and then to use it for trend removal. The typical representative here is Bjornland (Bjornland, 2000, p. 369). The second approach is frequency filtering. A number of filtrating methods have been developed to extract and eliminate non-stationarity and some of them have become very popular, e.g. the Hodrick-Prescott filter (Hodrick and Prescott, 1980), the band-pass filter of Baxter and King (Baxter and King, 1999), the filters of the Butterworth family (Gomez, 2001) and Wiener-Kolmogorov family (Planas, 1997). When being utilized these filters require subjective judgment at a particular moment. According to Stock and Watson (1996, p. 17), Kaiser and Maraval (1999) and Schenk-Hoppe (2001) this creates a serious problem - it becomes possible to generate spurious cycles in the filtered data even when there are no cycles. The latter is typical when the time series contains one or more unit roots (this is the Nelson and Kang (1981) critique).

Due to their ease of application, frequency filters have gained in popularity in recent years. Many papers adopted such techniques in the analysis of business cycle properties. It may happen that the results are sensitive to detrending methods and may lead to incorrect statements. The main goal of this paper is a partial examination of the problem with emphasis on possible generation of spurious cycles. The analysis covers the behavior of the filtered data and the influence of the ideal filters when presence of a unit root or a root near unity is possible. I shall analyze mainly the filter of Hodrick and Prescott and the band-pass filter in comparison with first-differencing.

The main conclusion from this paper is that application of ideal filters is dangerous when unit roots or near unit roots exist in time series. This can lead to detection of spurious cycles. A better solution would be a model-based approach for trend removal.

The paper is structured as follows. In part one the Hodrick-Prescott filter is analyzed. The band-pass filter is discussed in the second part. In part three "the ideal filter" is analyzed and compared with differencing. Conclusions are summarized in the last part.

Part One

Let's first consider the effects of the Hodrick-Prescott filter over a process containing a unit root. The filter is based on the assumptions that: (1) the tendency is smooth while the shocks have only temporary effect; (2) the process is linear. The first can be argued on the basis of the results reached by Nelson and Plosser (1982) and lately by Murray and Nelson (2000) and Fatas (2000). The second has been questioned since Hamilton (1989, 1990) developed his non-linear model with regime switching and subsequent analyses.

The Hodrick-Prescott filter assumes the presence of smooth trend independent of the cyclical component. Consider a time series y_t where t = 1, 2, ..., n and mark the tendency with g_t and the cyclical component – with c_t. We assume that the series is seasonally adjusted and irregular components play only a small role. Thus decomposition to unobserved components is:

$$y_t = g_t + c_t \tag{1}$$

Here the estimates of the tendency g_t and the cyclical component c_t are acquired after the minimization of

$$\Phi = \sum_{t=1}^{n} c_t^2 + \lambda \sum_{t=3}^{n} \left(\Delta^2 g_t\right)^2 \tag{2}$$

where Δ is the difference operator ($\Delta^k y_t = y_t - y_{t-k}$) and λ is a constant. The first sum accounts for the accuracy of the estimation (the residuals around the trend) while the second represents the smoothness of the trend. Parameter λ controls the weight of the two components, e.g. as we increase λ the HP trend becomes smoother and when λ approaches infinity the trend is transformed into a straight line.

The filter has the following advantages: it is easy to use; it is implemented in many computer packages; it allows clear interpretation of results and wide specification of the trend, that is up to the parabolic time trend and up to two unit roots.

But the filter also has serious drawbacks. Gomez (2001, p. 368) and Kaiser and Maraval (1999, p. 175) point out that the filtered series is subject to big changes when the raw data are revised or new data are added. The estimates are unstable, which is especially visible at the end of the series.

When the filter is used upon a series already subjected to seasonal adjustment, statistical inference could lead to incorrect results. As Kaiser and Maraval mention (1999, p. 184) the filter changes the autocorrelation structure of the process and spurious correlations can be spotted in the filtered data.

Gomes (2001, p. 368), Kaiser and Maraval (1999, p. 175) together with Stock and Watson (1996, p. 7) agree in their conclusions that the filter retains significant noise in the cyclical signal that hinders the accurate identification of the cyclical components.

King and Rebelo (1993) state that certain characteristic of the series and the filtered data are often different while Cogley and Nason (1995) show that when used upon a series with a unit root the filter produces series with autocorrelation at lag 1 as large as 0,72 no matter how large was the autocorrelation at lag 1 in the original data.

Finally there is the possibility of generating spurious cycles, as mentioned by Kaiser and Maraval (1999, p. 184). Gomez (2001, p. 368) concludes that the mechanical usage of the filter often leads to incorrect results and Schenk-Hoppe (2001, p. 83) directly states that the filter creates spurious cycles even when the original data contains no cycles at all.

When analyzing the effect of the combination of X11 and HP filter, Kaiser and Maraval performed a Monte Carlo simulation experiment and obtained two major results: (1) the period of the spurious cycles depends mainly on the order of the integration of the time series; (2) when the integration is fixed, the period depends on the value of λ (Kaiser and Maraval, 1999, pp. 189-190). Focusing mainly on the effect of the combination of the two filters (X11 and HP) they conclude that "the criticism that HP filtering induces a spurious cycle in the series is unwarranted" (p. 175) because the analyst "can choose the appropriate value of λ" (p. 191).

But it is the subjective choice of λ which contains the possibility of getting cycles with any period desired just by selecting the appropriate value of λ.

Schenk-Hoppe (2001) has experimented on a different ground. Based on the economic theory, he creates a model of an artificial economy and tracks down the trace of a priori determined tendency and cycles. Next he applies the HP and the BP filters to generated time series and compares both the real trend and cycles with those extracted by the filters. His results show that both filters were unable to trace the real cycles. Instead they extract spurious cycles with periods between 3 and 5 years. Schenk-Hoppe finishes his analysis with the conclusion that "any business cycle dating based on these filters leads to incorrect statements" (p. 83) and makes suggestions for the development of a new methodology or the modification of the existing one (p. 85).

The main problem is identified in the application of the filter over integrated series so I begin my analysis with this issue. To achieve my results I shall use the spectral analysis (see Hamilton, 1995, part 5, for example).

Let's have a linear stationary process x_t with the following ARMA representation:

$$\phi(L)x_t = \theta(L)e_t, \tag{3}$$

where $\phi(L)$ and $\theta(L)$ are the polynomials of the lag operator while e_t is a white noise. The spectral density of this process is given by:

$$Sx(\omega) = \frac{\sigma_e^2}{2\pi}\frac{\theta(e^{-i\omega})\theta(e^{i\omega})}{\phi(e^{-i\omega})\phi(e^{i\omega})} \tag{4}$$

where σ_e^2 is the variance of the white noise e_t.

In the presence of a unit root the process is no longer stationary and its spectrum is undefined. But it is still possible to derive the pseudo-spectrum, which is:

$$Sx(\omega) = \frac{\sigma_e^2}{2\pi}\frac{1}{1+\phi^2-2\phi\cos(\omega)} \tag{5}$$

where the dominating root is marked with ϕ and is not unity but approaches unity from below. When ϕ equals unity the statement becomes:

$$Sx(\omega) = \frac{\sigma_e^2}{4\pi}\frac{1}{1-\cos(\omega)} \tag{6}$$

which shows an infinite maximum in the spectral density at the zero frequency (the spectrum is not defined there).

To trace the effect of the application of the HP filter over a series containing a unit root I use the fact that the application of the filter over a series leads to new series with spectral density equal to the density of the original data multiplied by the transfer function of the filter called squared gain function.

The cyclical component extracted by the HP filter is given by:

$$C_t = \frac{\lambda(1-L)^2(1-L^{-1})^2}{1+\lambda(1-L)^2(1-L^{-1})^2}x_t \tag{7}$$

where L is the lag operator ($L^k x_t = x_{t-k}$).

Substituting L with $e^{-i\omega}$ I get the following gain function:

$$G(\omega) = \frac{\lambda(1-e^{-i\omega})^2(1-e^{i\omega})^2}{1+\lambda(1-e^{-i\omega})^2(1-e^{i\omega})^2} = \frac{4\lambda(1-\cos\omega)^2}{1+4\lambda(1-\cos\omega)^2} \tag{8}$$

whose square is the transfer function:

$$G^2(\omega) = \left[\frac{4\lambda(1-\cos\omega)^2}{1+4\lambda(1-\cos\omega)^2}\right]^2 \tag{9}$$

(The results up to here follow King and Rebelo, using the $e^{-ik\omega} = \cos k\omega - i.\sin k\omega$ equality and the appropriate transformations).

Now the application of the HP filter over a series x_t, which contains unit root:

$$z_t = C_{HP}(x_t) \tag{10}$$

leads to the following spectral density of the filtered series z_t:

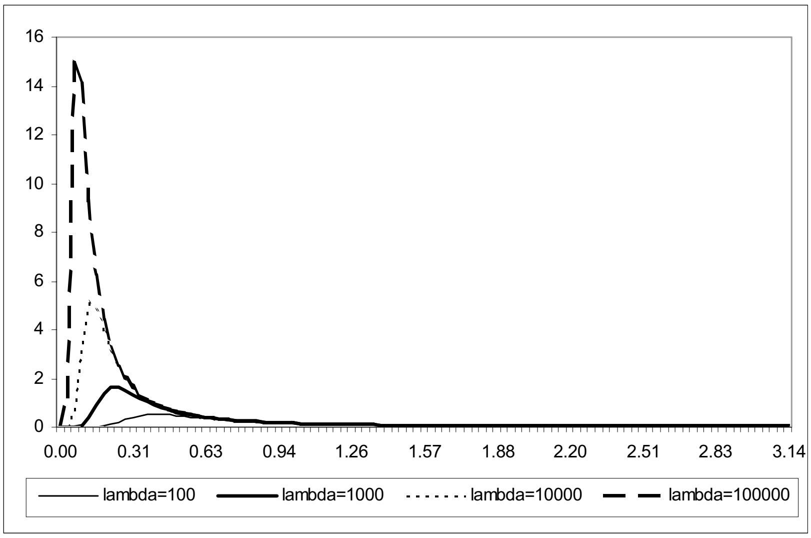
$$\begin{aligned} S_z(\omega) &= G^2(\omega).S_x(\omega) = \frac{\sigma_e^2}{4\pi}.\frac{1}{1-\cos\omega}.\frac{16\lambda^2(1-\cos\omega)^4}{\left[1+4\lambda(1-\cos\omega)^2\right]^2} = \\ &= \frac{4\sigma_e^2}{\pi}.\frac{\lambda^2(1-\cos\omega)^3}{\left[1+4\lambda(1-\cos\omega)^2\right]^2} = \frac{4\sigma_e^2}{\pi}.f(\lambda;\omega) \end{aligned} \tag{11}$$

where:

$$f(\lambda;\omega) = \frac{\lambda^2(1-\cos\omega)^3}{\left[1+4\lambda(1-\cos\omega)^2\right]^2} \tag{12}$$

The last result (11) shows that: (1) the spectrum is defined in the entire region $[0;\pi]$ contrary to the spectrum of the non-stationary series – it's a spectrum not pseudo-spectrum; (2) the spectral density at the zero frequency is zero – the HP filter cuts off the infinite maximum there. This is one of the features of the HP filter as mentioned above – the elimination of up to two unit roots. At the same time it is clear that the spectral density of the filtered data depends upon the value of parameter λ. The latter can be easily observed in figure 1.

Figure 1. Spectral density of the cyclical component when the HP filter is used over series with a unit root



The result shows that in the spectral density there is a maximum at the lower frequencies and it depends on the value of λ. Thus it is possible for false cycles to arise at low frequencies (business cycle frequencies). To determine the exact location of the maximum I further analyze function (12) for extremes because the other parameters in (11) are constants.

Treating λ as a constant the first derivative of (12) conditionally on ω is:

$$f'(\omega)=\left\{\frac{\lambda^2(1-\cos\omega)^3}{\left[1+4\lambda(1-\cos\omega)^2\right]^2}\right\}'=\frac{\lambda^2\sin\omega\left[3(1-\cos\omega)^2-4\lambda(1-\cos\omega)^4\right]}{\left[1+4\lambda(1-\cos\omega)^2\right]^3} \quad (13)$$

After equating to zero and solving the equation I get the following roots in the region $[0;\pi]$:

$$\omega_1=0;$$
$$\omega_2=\pi$$
$$\omega_3=\cos^{-1}\left(1-\frac{\sqrt{3}}{2\sqrt{\lambda}}\right)$$

where $\cos^{-1}$ is the inverse cosine function.

The values of the spectral density in the first two points (which are at the ends of the region) are 0 and

$$\frac{\sigma_e^2}{\pi}\cdot\frac{32\lambda^2}{(1+16\lambda)^2}$$

respectively, while the third one is inside the interval and values

$$\frac{\sigma_e^2}{\pi}\cdot\frac{3\sqrt{3\lambda}}{32}.$$

As the third value is the biggest for any positive value of λ the maximum is achieved at that point. Thus if the HP filter is applied over a time series with a unit root the result is a series with cyclical fluctuations at the frequency

$$\omega = \cos^{-1}\left(1-\frac{\sqrt{3}}{2\sqrt{\lambda}}\right).$$

This finding supports the simulation results of Kaiser and Maraval (1999, p. 190), as there is almost perfect fit of theoretical values from the formula with the ones produced by the simulation. The two authors state that "the relationship between this period and λ is highly nonlinear". The relationship is indeed nonlinear although not very complicated. If the values for λ, suggested by Hodrick and Prescott (144000 for monthly data and 1600 for quarterly data) are put into the formula the results show that there will be maximums at 92,98 months and 30,14 quarters respectively. Both cases are between 7 and 8 years. It is not surprising that the application of the filter in many empirical studies leads to cycles with such or nearly the same period. This is because many economic time series either contain unit root or at least unit root offers good approximation to the true form of non-stationarity.

At the next stage we are interested in the first autoregressive root that is less than unity in modulus. Let's consider the simplest case with an AR(1) process:

$$y_t = \phi y_{t-1} + e_t \tag{14}$$

where ϕ is less than 1 but positive.

The spectral density of the process (14) is given by:

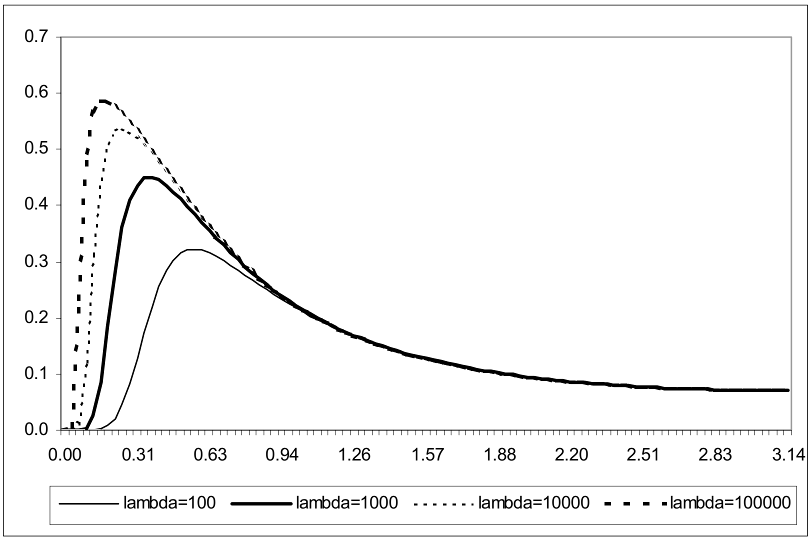
$$S_y(\omega) = \frac{\sigma_e^2}{2\pi}\frac{1}{1+\phi^2-2\phi\cos(\omega)} \tag{15}$$

The application of the HP filter leads to cyclical component z_t with the following spectral density:

$$S_z(\omega) = G^2(\omega).S_y(\omega) = \frac{\sigma_e^2}{2\pi}.\frac{1}{\left[1+\phi^2-2\phi\cos\omega\right]}.\frac{16\lambda^2(1-\cos\omega)^4}{\left[1+4\lambda(1-\cos\omega)^2\right]^2} \quad (16)$$

The image of the spectral density for ϕ = 0.5 and different values of λ can be seen in figure 2. Again a maximum exists at the low frequencies.

Figure 2. Spectral density of the cyclical component when the HP filter is used over series with first autoregressive root of 0.5



Unfortunately the exact functional form of the relationship between the frequency at the maximum and the parameters ϕ and λ cannot be derived analytically. After tedious computations the first derivative turns out to be an equation of fifth power. A better solution is to use numerical methods to compute both the maximum and the correspondent frequency for any given values of ϕ and λ. This does not change the situation because the shape of the spectral density and the conclusions remain the same. Spurious cyclical fluctuations are generated by the HP filter not only when the first root is unity but also when the fist autoregressive root is large. The maximums in spectral density disappear only when the value of ϕ becomes negligibly small. The reason for such a behavior is the form of the spectral density of the original process, which is decreasing. The HP filter cuts off part of it around the zero frequency but the rest remains and generates an artificial maximum.

Part II

The next filter is band-pass (BP) filter of Baxter and King. It's a symmetric moving average filter with weights that sum to zero. It ensures the elimination of the infinite maximum at the zero frequency as shown by Baxter and King (1999, p. 592, Appendix A). In their paper Baxter and King developed the weights for three cases: two for monthly data and one for yearly data. Because of the time consuming computations I derive the spectral function analytically only for the third case, with weights of –0.0510, –0.1351, –0.2010, 0.7741, –0.2010, -0.1351 and -0.0510 (p. 591, Table 4, Column 4).

The filtered series (pure cyclical component) is given by:

$$z_t = \phi\,(L)x_t \tag{17}$$

where the filter is:

$$\phi(L) = -\,0.0510L^{-3} - 0.1351L\text{-}2 - 0.2010L\text{-}1 + 0.7741L^{0} - \\ - 0.2010\mathrm{L}^{-1} - 0.1351L^{2} - 0.0510L^{3} \tag{18}$$

Substituting L with $e^{-i\omega}$ and simplifying I get the transfer function:

$$G^2(\omega) = \left(0.7741 - 0.1020\cos 3\omega - 0.2702\cos 2\omega - 0.4020\cos\omega\right)^2 \tag{19}$$

The application of the BP filter over series with a unit root leads to the following spectral density:

$$S_z(\omega) = S_x(\omega).G^2(\omega) = \frac{\sigma_e^2}{4\pi}.\frac{\left(0.7741 - 0.1020\cos 3\omega - 0.2702\cos 2\omega - 0.4020\cos\omega\right)^2}{1-\cos\omega}$$

$$S_z(\omega) = \frac{\sigma_e^2}{4\pi}.(1.0908 + 0.8902\cos\omega - 0.2293\cos^2\omega - \\ - 0.9778\cos^3\omega - 0.6074\cos^4\omega - 0.1665\cos^5\omega) \tag{20}$$

After computing the first derivative and equating to zero I get the next three real solutions in the region $[0;\pi]$:

$$\omega_1 = 0;$$
$$\omega_2 = \pi;$$
$$\omega_3 = 1.143774.$$

Again the first two are at the ends of the interval with values of 0 and

$$0.5079.\frac{\sigma_e^2}{4\pi}$$

The third one is inside the region and there the spectral density reaches its maximum of

$$1.3305.\frac{\sigma_e^2}{4\pi}$$

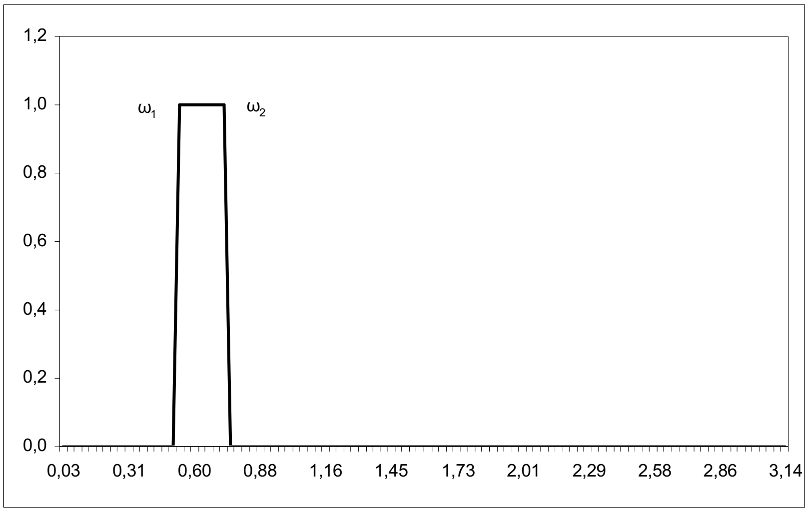
This frequency corresponds to cycles with period of 5.49 years.

Part III

The Hodrick-Prescott filter can be derived from both the Wiener-Kolmogorov and the Butterworth families (Planas, 1997; Gomez, 2001). I shall not analyze the filters from these families as they depend on many more parameters. Instead I shall analyze the ideal filter and show that the conclusions derived in previous parts still hold.

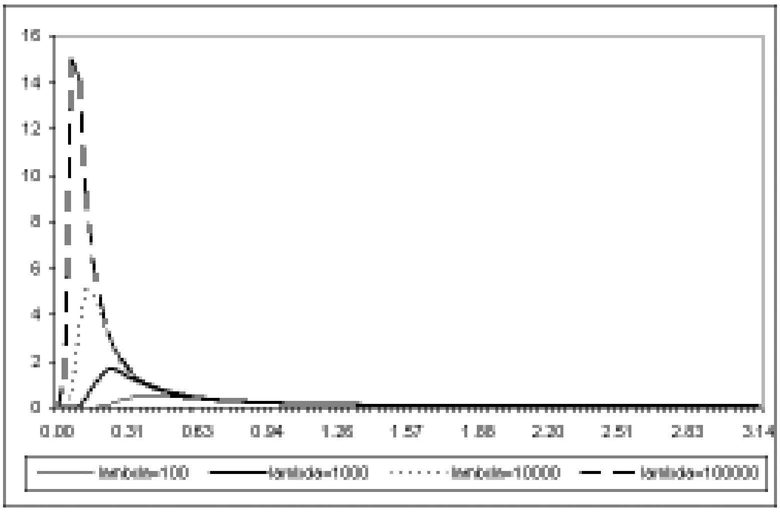
Baxter and King define the ideal filter as a filter that keeps unchanged the spectral density in some interval [ω_1; ω_2] while making density at all the other frequencies nearly zero. Usually the researcher defines this interval. The graphical form of the ideal filter is represented in figure 3, where its transfer function is shown – it values 1 for the desired interval and zero elsewhere.

Figure 3. Transfer function of the ideal filter



To get an idea of the effect of such a filter over unit root processes we look at the resulting spectrum at figure 4.

Figure 4. Result from the application of the ideal filter over the series with a unit root.

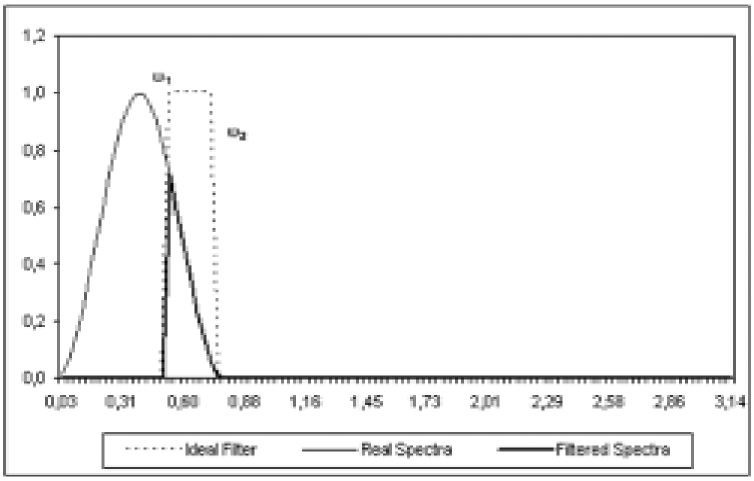


Only the part lying between the two frequencies ω_1 and ω_2 is transferred. Because the spectrum of a non-stationary process decreases in the entire region $[0;\pi]$ its part lying in the $[\omega_1; \omega_2]$ interval also decreases. Thus there will be a maximum at the beginning of the interval at ω_1 frequency. The application of the ideal filter over a unit root process will lead to the registration of artificial cycles at frequency ω_1. Again it is the researcher who selects this frequency, so he selects the period of the cycles, too. With the HP filter this is a selection of the value of λ while with the BP filter it is a selection of the filter's weights. As approximations of the ideal filter these lead to the same results. If the cause is in the non-stationarity, it is advisable to remove the non-stationarity first, as suggested by Bjornland. He proposes to analyze the series first and then to select the best transformation on the basis of the series' dynamic properties. But the real problem is not in the unit root because the result doesn't change when the first root is smaller than unity. The difference is in the value of the maximum, which decreases with the decline of ϕ in modulus. But both the shape of the spectral density and the conclusions remain the same. The real problem here is the shape of transfer function of the ideal filter, which contains a maximum in the interval $[\omega_1; \omega_2]$.

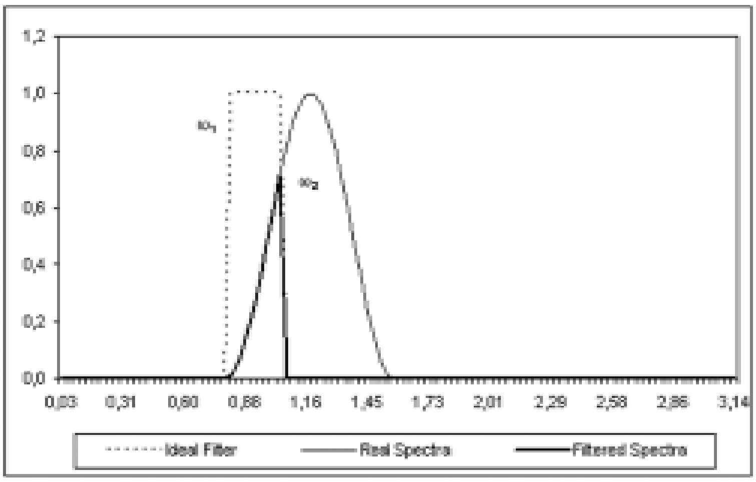
To strengthen that conclusion let's consider the case when in the original data there are cycles at frequency ω_0 near but outside the interval $[\omega_1; \omega_2]$. In such a situation the ideal filter will transfer cycles from the real frequency ω_0 to the lowest possible frequency ω_1 if $\omega_0 < \omega_1$ or to the highest possible ω_2 if $\omega_0 > \omega_2$. The latter is clearly seen in figure 5.

Figure 5. Result from the application of the ideal filter over the series containing cycles outside the region of interest.

(a)



(b)



Thus the application of the "ideal filter" leads to spurious cycles not only in the presence of non-stationarity but also in the case of stationary data with cycles. Any filter that approximates the ideal filter leads to the same result, i.e. artificial cycles with a period determined by the researcher. To overcome this problem it is recommended that we analyze series for the trends and the cycles and only then apply the filter. Naturally after such an analysis the application of the ideal filter is useless.

To remove the non-stationarity Box, Jenkins and Reinsel (1994) suggested the use of the difference operator. It can be considered as a filter like those already examined, for the purpose of comparison. Baxter and King compare it with their filter and determine its main drawbacks in two directions. First, they conclude that the filter is not symmetric which can cause problems if applied to more than one variable linked between them. The filter generates phase changes in the spectrums of the variables and the existing relationships can be diluted. Second, a critique also supported by Hodrick and Prescott, Maraval, Gomez, the filter drastically changes the power of the spectral density by frequencies. It gives bigger weight to the high frequencies while decreasing the power of the low and eliminating completely the zero frequency. The conclusion of Baxter and King is that "If the goal of a business-cycle filter is to isolate fluctuations in the data that occur between specific periodicities, without special emphasis on any particular frequency, the first-difference filter is a poor choice." (pp. 585 - 586) In my opinion such a conclusion is not right.

Let's first consider the transfer function of the difference filter, as did Baxter and King. The function is:

$$G^2(\omega) = \left(1 - e^{-i\omega}\right)\left(1 - e^{i\omega}\right) = 2\left(1 - \cos\omega\right) \tag{21}$$

and indicates the extent to which the filter raises (lowers) the variance of the filtered series relative to that of the original series.

As can be seen the influence of the filter is like that described above, i.e. the variance at high frequencies is expanded four times while at the low ones it is decreased. But this is the influence when the filter is applied over stationary series while the main purpose of the filter is to be applied over non-stationary ones.

As correctly mentioned by den Haan (pp. 9 - 10) when comparing the results from the different filters the researchers usually compare different things. Here the application of the difference filter over stationary series (where it gives the worst results) is compared with the results of the filter described (HP and BP) when applied over stationary series (they are created for such a purpose). Let's first compare the performance of the filters over non-stationary series using the results from the previous parts.

To see the effect of the difference filter let's consider a linear stochastic process from the *ARIMA* family: *ARIMA* (*p, d, q*), where $d = 1$ for simplicity. The process can be presented as:

$$(1-L)\phi(L)y_t = \theta(L)\varepsilon_t \tag{22}$$

with a spectral density:

$$S_y(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \cdot \frac{1}{2(1-\cos\omega)} \cdot \frac{\theta(e^{-i\omega})\theta(e^{i\omega})}{\phi(e^{-i\omega})\phi(e^{i\omega})} \tag{23}$$

It's obvious that the spectrum is not defined at the zero frequency. This is pseudo-spectrum. The application of the first-difference filter:

$$x_t = (1-L)y_t \tag{24}$$

leads to a series with the following spectral density:

$$S_x(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \cdot \frac{1}{2(1-\cos\omega)} \cdot \frac{\theta(e^{-i\omega})\theta(e^{i\omega})}{\phi(e^{-i\omega})\phi(e^{i\omega})} .2(1-\cos\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{\theta(e^{-i\omega})\theta(e^{i\omega})}{\phi(e^{-i\omega})\phi(e^{i\omega})} \tag{25}$$

But this is exactly the spectral density of the *ARMA* (*p, q*) process, i.e. the stationary part of the original process. No increase of the high frequencies or decrease of the low ones is observed. The filter just removes the influence of the unit root. It removes the tendency not only at the zero frequency as the other filters do but also removes the influence of the tendency in both the low frequencies (where the tendency has high influence) and the high ones (where the effect of the tendency is negligible).

If the series looks rough after such a transformation it's because the tendency smoothes the series. It can be concluded that the main argument that the filter increases the high frequencies is not consistent because it should be used only when a unit root is present or such an assumption cannot be rejected.

The situation is different when the dominating root is less then unity. In this case the spectral density of the original data is:

$$S_y(\omega) = \frac{\sigma_e^2}{2\pi} \frac{\theta(e^{-i\omega})\theta(e^{i\omega})}{\phi(e^{-i\omega})\phi(e^{i\omega})} \tag{26}$$

and the density of the filtered series is:

$$S_x(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \cdot \frac{\theta(e^{-i\omega})\theta(e^{i\omega})}{\phi(e^{-i\omega})\phi(e^{i\omega})} .2(1-\cos\omega) \tag{27}$$

The difference filter removes the maximum at zero and quadruples the high frequencies variance.

The second critique was about the phase shift due to the asymmetry of the difference filter. I do not think this will generate problems in practice. If more than one series are analyzed and the filter is applied to all of them, the phase shift will be present in all series and the existing relationships will remain. The potential danger lies in the possibility of applying the filter only in part of the series while the others (which are stationary) are analyzed without transformation. Regarding the ideas about cointegration and the relationships between integrated processes, the possibility of relationships between stationary and integrated variables is very unlikely. Thus the second critique is not problematic.

Part IV

I can summarize the following propositions:

First, when a unit root is present in the data first-differencing results in a stationary process while "ideal" filters lead to spurious cycles.

Second, when the process is stationary differencing boosts the high frequencies. On the other hand, ideal filters will either generate spurious cycles when a significant first autoregressive root exists or may transfer cycles if they are present.

The danger of false cycles exists in both cases when applying the ideal filter. From this point of view the application of differencing is recommended especially when combined with the appropriate testing for unit roots. If we have a stationary series, but due to the low power of the unit root tests we accept that the series is a non-stationary one, the application of the filter will intensify the high frequencies. But such a decision is not as dangerous as the case where missing accommodation for the unit root will generate spurious cycles.

In this paper we used first-differencing as an example of detrending when non-stationarity takes a particular form - e.g. unit root. In general, series may be non-stationary but not with a unit root (fractional integration, polynomial trends). As long as there is large autoregressive root in the ARIMA representation of the process the application of the ideal filter will generate false cycles. Instead of frequency filtering or differencing other trend removal procedures may be needed in accordance with the data generating process. The main requirement for the transfer function of the procedure is to be monotonic. Thus the possibility of generating spurious cycles is eliminated unconditionally because transfer function contains no extremes.

Another possible solution is to use both methods (filtering and differencing) at the same time. If after the trend removal the spectral estimates differ only in their amplitudes in the high frequency region, no problems exist with the application of the ideal filters. But if the methods record different cycles, the series probably contains a unit root, a large positive autoregressive root or cycles at frequency near the region of interest. Situations like these are often met in practice and some research papers definitely need reexamination because the near ideal filters have been used.

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